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The stiffness of quasilinear gyroscopically coupled systems with respect to low and high frequency periodic perturbations☆

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Abstract

The stability of the equilibrium of gyroscopically coupled quasilinear systems with many degrees of freedom is investigated when there is dissipation and a periodic perturbation which is not necessarily of small amplitude. Non-potential forces (customarily referred to as radial correction forces or circulating forces) act together with potential forces. Under conditions of a low- and high-frequency periodic perturbation, classes of systems are distinguished using Lyapunov functions which possess the property of unperturbability, that is, their qualitative structure remains almost the same as in the case of autonomous systems. Generalizations to the case of non-periodic perturbations are possible.

The problem of the stability of an equilibrium when gyroscopic forces act goes back to the time of Thomson and Tait[1] and was subsequently studied by Routh,[2] Chetayev[3] and many of their successors. While, in the classical papers, gyroscopic stabilization was considered for cases when the positional forces are potential, the subsequent generalizations also take into account positional non-potential forces.[4–8] Parametrically perturbable systems are considered below as an extension of the preceding investigation,[9] where both the gyroscopic constraint as well as the positional non-potential forces are taken into account. For these systems, a threshold of non-perturbability is established using Lyapunov functions in cases when the corresponding frequency of the periodic perturbation may tend both to zero and to infinity.

1. Introduction

Consider the non-autonomous system with n degrees of freedom

$$A_0\ddot{\mathbf{q}} + [D + G(\omega t)]\dot{\mathbf{q}} + [\Pi(\omega t) + C(\omega t)]\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})$$
$$A_0 = A_0^T, \quad \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C^{1,1,1}_{t\mathbf{q},\dot{\mathbf{q}}}(R \times D_{\mathbf{q}\dot{\mathbf{q}}}), \quad \|\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})\| = o(\|\mathbf{q}, \dot{\mathbf{q}}\|) \tag{1.1}$$

where A_0 is a constant matrix with positive eigenvalues and the matrices $G(\omega t)$, $\prod(\omega t)$, $C(\omega t) \in C^1_t(R)$ and the vector $\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})$ depend $(2\pi/\omega)$-periodically on t $(\omega > 0)$. We shall assume that the matrix D is constant and that $D = D^T$.

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Since dissipation is usually small, the representation of the matrix D in the form $D=\alpha D_0$, where α is a small positive parameter, is then permitted and the matrix D_0 has positive eigenvalues. The matrices $G(\omega t)$, $\prod(\omega t)$ and $C(\omega t)$ have the following structure:

$$G(\omega t) = G_0 + G_1(\omega t), \quad G^T = -G$$
$$\Pi(\omega t) = \Pi_0 + \Pi_1(\omega t), \quad \Pi^T = \Pi \tag{1.2}$$
$$C(\omega t) = C_0 + C_1(\omega t), \quad C^T = -C$$

where G_0, $\prod_0$, C_0 are constant matrices. Since $G(\omega t)$, $\prod(\omega t)$, $C(\omega t) \in C_t^1(R)$, the norms of both the matrices $G_1(\omega t)$, $\prod_1(\omega t)$, $C_1(\omega t)$ and of their derivatives are bounded.

Next, without loss of generality, we shall assume that all of the coefficients of system (1.1) as well as the time t are dimensionless quantities.

We will investigate the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=0$ of system (1.1) with the assumptions that have been made regarding it.

2. The stability of the equilibrium of system (1.1) under conditions of a low-frequency periodic perturbation

Starting out from the linear approximation of system (1.1), we will denote the roots of the equation

$$\left|A_0\lambda^2 + [D + G(\omega t)]\lambda + [\Pi(\omega t) + C(\omega t)]\right| = 0 \tag{2.1}$$

by $\lambda_1(t), \ldots, \lambda_{2n}(t)$ and introduce the matrices

$$A_0^{-1}[D + G(\omega t)] = \hat{G}(\omega t), \quad A_0^{-1}[\Pi(\omega t) + C(\omega t)] = \hat{C}(\omega t)$$

Theorem 1. *Suppose the matrices $\hat{G}(\omega t)$ and $\hat{C}(\omega t)$ contain only the parameter ω as a factor accompanying t.*

Then a threshold value of the frequency $\omega=\omega_0$ exists such that, when $\omega<\omega_0$, the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=0$ of system (1.1) is asymptotically stable if

$$\mathrm{Re}\lambda_i(t) < -\gamma_1, \quad \mathrm{const} = \gamma_1 > 0, \quad i = 1, 2, \ldots, 2n \tag{2.2}$$

and, on the other hand, unstable if just a single root $\lambda^*(t)$ of Eq. (2.1) exists such that

$$\mathrm{Re}\lambda^*(t) > \gamma_2, \quad \mathrm{const} = \gamma_2 > 0 \tag{2.3}$$

Proof. We will assume that condition (2.2) is satisfied. Then, according to Lyapunov's theorem[10] (see also Ref. 11) a positive-definite quadratic form

$$V = \frac{1}{2}\dot{\mathbf{q}}^T A_1(\omega t)\dot{\mathbf{q}} + \mathbf{q}^T B_1(\omega t)\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T D_1(\omega t)\mathbf{q}, \quad V(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C_t^1(R) \tag{2.4}$$

exists which satisfies the equation

$$L(V) = U(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \tag{2.5}$$

in which the Lyapunov operator $L(V)$ is defined by the equality

$$L(V) = [\dot{\mathbf{q}}^T A_1(\omega t) + \mathbf{q}^T B_1(\omega t)][-\hat{G}(\omega t)\dot{\mathbf{q}} - \hat{C}(\omega t)\mathbf{q}] + \dot{\mathbf{q}}^T B_1(\omega t)\dot{\mathbf{q}} + \dot{\mathbf{q}}^T D_1(\omega t)\mathbf{q} \tag{2.6}$$

and we choose the negative-definite quadratic form $U(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C_t^1(R)$ to depend $(2\pi/\omega)$-periodically on t and to contain only the parameter ω as a factor accompanying t.

In order to find the coefficients of the form V, according to Eq. (2.5) we obtain a system of $n(2n+1)$ linear inhomogeneous equations. On the basis of Eq. (2.5), the elements of the matrices $A_1(\omega t)$, $B_1(\omega t)$, $D_1(\omega t)$, as solutions of the corresponding system of linear equations, can be represented in the form

$$a_{1ij}(\omega t) = \frac{a^0_{1ij}(\omega t)}{d(\omega t)}, \quad b_{1ij}(\omega t) = \frac{b^0_{1ij}(\omega t)}{d(\omega t)}, \quad d_{1ij}(\omega t) = \frac{d^0_{1ij}(\omega t)}{d(\omega t)}; \quad i, j = 1, 2, \ldots, n$$

where $d(\omega t) \in C^1_t(R)$ is a $n(2n+1)$- order determinant which is composed of the coefficients of the linear approximation of system (1.1), and the quantities $a^0_{1ij}(\omega t)$, $b^0_{1ij}(\omega t)$, $d^0_{1ij}(\omega t) \in C^1_t(R)$ depend both on the coefficients of the linear approximation of system 91.1) and, also, on the coefficients of the form $U(\omega t, \mathbf{q}, \dot{\mathbf{q}})$. According to the conditions of Theorem 1, the determinant $d(\omega t)$ satisfies the inequality

$$|d(\omega t)| \geq d_0, \quad 0 < d_0 = \text{const}$$

and, consequently, $a_{1ij}(\omega t)$, $b_{1ij}(\omega t)$, $d_{1ij}(\omega t) \in C^1_t(R)$.

We will use the quadratic form V as the Lyapunov function. Calculating the derivative of the function V with respect to t along the vector field, which is defined by system (1.1), we obtain

$$\frac{dV}{dt} = U(\omega t, \mathbf{q}, \dot{\mathbf{q}}) + \omega\frac{\partial V}{\partial \omega t} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2) \tag{2.7}$$

where

$$\frac{\partial V}{\partial \omega t} = \frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial A_1(\omega t)}{\partial \omega t}\dot{\mathbf{q}} + \mathbf{q}^T\frac{\partial B_1(\omega t)}{\partial \omega t}\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T\frac{\partial D_1(\omega t)}{\partial \omega t}\mathbf{q}$$

The quadratic form $\partial V/\partial \omega t$ has bounded coefficients. Hence, when the parameter ω tends to zero in equality (2.7), we arrive at a conclusion concerning the existence of a threshold value ω_0 of the parameter ω such that, when $\omega < \omega_0$, the right-hand side of this equality becomes negative-definite. From this, we conclude that Theorem 1 is correct in the case of condition (2.2).

Now suppose a root $\lambda^*(t)$ of Eq. (2.1) exists such that condition (2.3) is satisfied. Then, according to Lyapunov's theorem, a quadratic form with alternating signs

$$W = \frac{1}{2}\dot{\mathbf{q}}^T A_2(\omega t)\dot{\mathbf{q}} + \mathbf{q}^T B_2(\omega t)\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T D_2(\omega t)\mathbf{q}, \quad W(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C^1_t(R)$$

exists which satisfies the equation

$$L(W) = \kappa W + U_1(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \tag{2.8}$$

in which the Lyapunov operator $L(W)$ is determined by an equality of the form (2.6) with the matrices $A_1(\omega t)$, $B_1(\omega t)$, $D_1(\omega t)$ in it replaced by the matrices $A_2(\omega t)$, $B_2(\omega t)$, $D_2(\omega t)$. On the right-hand side of equality (2.8), κ is a positive constant, and we choose the positive-definite quadratic form $U_1(\omega_t, \mathbf{q}, \dot{\mathbf{q}}) \in C^1_t(R)$ to depend $(2\pi/\omega)$- periodically on t and to contain only the parameter ω as a factor accompanying t.

Determining the coefficients of the quadratic form W in an analogous manner to that described above, we use this quadratic form as the Lyapunov function. Calculating the derivative of W with respect to t along the vector field determined by system (1.1), we have

$$\frac{dW}{dt} = \kappa W + U_1(\omega t, \mathbf{q}, \dot{\mathbf{q}}) + \omega\frac{\partial W}{\partial \omega t} + o(\|\mathbf{q}, \dot{\mathbf{q}}\|^2) \tag{2.9}$$

Next, following the scheme described above, we conclude that a threshold value ω_0 of the parameter ω exists such that, when $\omega < \omega_0$, the function

$$U_1(\omega t, \mathbf{q}, \dot{\mathbf{q}}) + \omega\frac{\partial W}{\partial \omega t}$$

becomes positive-definite. Therefore, in accordance with equality (2.9), when account is taken of the sign alternation of the quadratic form W we conclude that all the conditions of the Chetayev instability theorem hold. □

Theorem 1 is proved.

So, under the conditions of Theorem 1, the behaviour of system (1.1) is completely determined by the properties of the roots of Eq. (2.1), that is, we have a situation which is very close to the case of autonomous systems when the coefficients of a system do not change with time.

Example 1. Consider the system[9]

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + B(\omega t)\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}), \quad \|\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})\| = o(\|\mathbf{q}, \dot{\mathbf{q}}\|) \tag{2.10}$$

in which α is a small positive parameter and the matrix D_0 has positive eigenvalues. The matrix $B(\omega t)$ admits of the representation $B(\omega t) = B_0 + \beta B_1(\omega t)$, where the parameter β is so small that the eigenvalues $b_j(t)$ of the matrix $B(\omega t)$ are positive and

$$b_j(t) \ge \tilde{b}_j, \quad 0 < \tilde{b}_j = \text{const}, \quad j = 1, 2, \dots, n$$

It can be shown that, in this case, condition (2.2) is satisfied for the roots of the equation

$$\left|A_0\lambda^2 + \alpha D_0\lambda + B(\omega t)\right| = 0$$

We shall assume that the matrix $B(\omega t)$ only contains the parameter ω as a factor accompanying t.

Choosing the function V in the form of (2.4), we consider the equation

$$\begin{aligned}&[\dot{\mathbf{q}}^T A_1(\omega t) + \mathbf{q}^T B_1(\omega t)][-A_0^{-1}\alpha D_0\dot{\mathbf{q}} - A_0^{-1}B(\omega t)\mathbf{q}] + \\ &+ \dot{\mathbf{q}}^T B_1(\omega t)\dot{\mathbf{q}} + \dot{\mathbf{q}}^T D_1(\omega t)\mathbf{q} = -\dot{\mathbf{q}}^T(\alpha D_0 - \gamma A_0)\dot{\mathbf{q}} - \gamma\mathbf{q}^T B(\omega t)\mathbf{q}\end{aligned} \tag{2.11}$$

as Eq. (2.5), where the positive constant $\gamma < \alpha\lambda^-$, and λ^- denotes the smallest characteristic number of the equation

$$|D_0 - \lambda A_0| = 0$$

Eq. (2.11) is satisfied if

$$A_1(\omega t) = A_0, \quad B_1(\omega t) = \gamma A_0, \quad D_1(\omega t) = [B(\omega t) + \alpha\gamma D_0]$$

and the function V in the case being considered is therefore determined by the expression

$$V = \frac{1}{2}\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}} + \gamma\mathbf{q}^T A_0\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T[B(\omega t) + \alpha\gamma D_0]\mathbf{q} \tag{2.12}$$

The derivative of the function V with respect to t along the vector field defined by system (2.10) has the form

$$\frac{dV}{dt} = -\dot{\mathbf{q}}^T(\alpha D_0 - \gamma A_0)\dot{\mathbf{q}} - \gamma\mathbf{q}^T B(\omega t)\mathbf{q} + \frac{\omega}{2}\mathbf{q}^T\left[\beta\frac{\partial B_1}{\partial(\omega t)}\right]\mathbf{q} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2) \tag{2.13}$$

It follows from equality (2.13) that, for a sufficiently small value of ω, the derivative dV/dt becomes negative-definite. Actually, if the choice of the constant γ satisfies the condition

$$\beta\mu^+\omega/2 < \gamma < \alpha\lambda^- \tag{2.14}$$

where the number μ^+ corresponds to $\sup(\mu_1(t), \dots, \mu_n(t))$ and the quantities $\mu_i(t)$ are the characteristic numbers of the equation

$$|\partial B_1/\partial(\omega t) - \lambda B(\omega t)| = 0$$

then the derivative dV/dt is negative-definite. In turn, condition (2.14) makes sense if $\beta\mu^+\omega/2 < \alpha\lambda^-$ and therefore

$$\omega < \omega_0 = 2\alpha\lambda^-/(\beta\mu^+)$$

Consequently, the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = 0$ of system (2.10) is asymptotically stable if $\omega < \omega_0$ and the eigenvalues of the matrix $B(\omega t)$ are positive and detached from zero

Remark 1. In the example which has been considered, the threshold value ω_0 of the parameter ω has been determined, starting out from the structure of the corresponding equations, which ensures the asymptotic stability of the equilibrium. Coarser estimates of ω can be directly derived from the scheme of proof of Theorem 1.

We will now consider the regular bundle of quadratic forms

$$\frac{\partial V}{\partial \omega t} - \lambda[-U(\omega t, \dot{\mathbf{q}}, \mathbf{q})] \tag{2.15}$$

Suppose the number λ^+ corresponds to $\sup(\lambda_1(t), \ldots, \lambda_{2n}(t))$, where $\lambda_i(t)$ are the characteristic numbers of the bundle of forms being considered.

We represent equality (2.7) in the form

$$\frac{dV}{dt} = \omega\left\{\frac{\partial V}{\partial \omega t} - \omega^{-1}[-U(\omega t, \dot{\mathbf{q}}, \mathbf{q})]\right\} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2) \tag{2.16}$$

Comparing the bundle (2.15) and the right-hand side of equality (2.16), we conclude that the derivative dV/dt is negative-definite if

$$\omega^{-1} > \lambda^+$$

whence $\omega < 1/\lambda^+$ and, therefore, $\omega_0 = (\lambda^+)^{-1}$.

When instability is discussed, we consider the regular bundle of quadratic forms in the form

$$-\frac{\partial W}{\partial \omega t} - \lambda U_1(\omega t, \dot{\mathbf{q}}, \mathbf{q})$$

representing equality (2.9) as follows:

$$\frac{dW}{dt} = \kappa W - \omega\left[-\frac{\partial W}{\partial \omega t} - \omega^{-1} U_1(\omega t, \dot{\mathbf{q}}, \mathbf{q})\right] + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)$$

Then by arguing in a similar manner as described above, we determine ω_0.

Remark 2. It can happen that the representation

$$G_1(\omega t) = \omega G_1^*(\omega t), \quad \Pi_1(\omega t) = \omega \Pi_1^*(\omega t), \quad C_1(\omega t) = \omega C_1^*(\omega t)$$

holds in equalities (1.2).

Then, the condition of Theorem 1, which concerns the way in which the parameter ω occurs in the equations of motion (1.1), is not satisfied. However, there is no difficulty in investigating the stability, in this case also by formulating the corresponding requirements with respect to the roots of the equation

$$|A_0\lambda^2 + (D + G_0)\lambda + (\Pi_0 + C_0)| = 0$$

We will later use the Lyapunov function, constructed for the unperturbed system

$$A_0\ddot{\mathbf{q}} + [D + G_0]\dot{\mathbf{q}} + [\Pi_0 + C_0]\mathbf{q} = \mathbf{0} \tag{2.17}$$

to investigate the complete equations of motion (1.1). In particular, in the case being considered, the equalities

$$\frac{dV_1}{dt} = U(\dot{\mathbf{q}}, \mathbf{q}) - \omega W_1 + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)$$

$$\frac{dV_2}{dt} = \kappa V_2 + U_1(\dot{\mathbf{q}}, \mathbf{q}) - \omega W_2 + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)$$

where

$$W_i = \{\dot{\mathbf{q}}^T A_i A_0^{-1} G_1^*(\omega t)\dot{\mathbf{q}} + \mathbf{q}^T[B_i A_0^{-1} G_1^*(\omega t) + (\Pi_1^*(\omega t) - C_1^*(\omega t))A_0^{-1}A_i]\dot{\mathbf{q}} + $$
$$+ \mathbf{q}^T B_i A_0^{-1}[\Pi_1^*(\omega t) + C_1^*(\omega t)]\mathbf{q}\}, \quad i = 1, 2$$

and the functions

$$V_i = \frac{1}{2}\dot{\mathbf{q}}^T A_i \dot{\mathbf{q}} + \mathbf{q}^T B_i \dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T D_i \mathbf{q}$$

correspond to the unperturbed system (2.17), are analogues of equalities (2.7) and (2.9).

On combining the approach proposed here with the scheme for the proof of Theorem 1, there is no difficulty in treating the situation when only some of the coefficients of the matrices $G_1(\omega t)$, $\prod_1(\omega t)$, $C_1(\omega t)$ are multiplied by ω or by positive powers of ω. Within the framework of this approach, only the case when the coefficients of the above matrices are multiplied by negative powers of ω is inadmissible.

Example 2. We will investigate the stability of a gyro-horizon compass in the case of the circulation of a ship.[12] The corresponding system of equations of motion can be reduced to the form[12]

$$\ddot{\boldsymbol{\xi}} + 2b\dot{\boldsymbol{\xi}} + \nu^2\boldsymbol{\xi} + C(\omega t)\boldsymbol{\xi} = \mathbf{O}(\dot{\boldsymbol{\xi}}^2 + \boldsymbol{\xi}^2), \quad \boldsymbol{\xi} = (\xi_1, \xi_2)^T \tag{2.18}$$

where

$$C(\omega t) = \begin{pmatrix} 0 & 2b\Omega \\ -2b\Omega & 0 \end{pmatrix}, \quad \Omega = \mp\mu\omega\sin(\psi_0 \pm \omega t)$$

and the positive constant b corresponds to a damping fautor, ν is the Schuler frequency, Ω is the projection of the absolute angular velocity of the sensitive component of the gyrocompass on to the geocentric vertical of the position, μ is a small parameter, ω is the angular velocity of circulation, and Ψ_0 is the initial track. We therefore have the situation considered in Remark 2.

We now consider the unperturbed system

$$\ddot{\boldsymbol{\xi}} + 2b\dot{\boldsymbol{\xi}} + \nu^2\boldsymbol{\xi} = \mathbf{0}$$

It is quit obvious that its equilibrium position is asymptotically stable. Consequently, a Lyapunov function exists which satisfies Lyapunovr's theorem on asymptotic stability. It is easy to show that this function is a quadratic form of the form (2.12)

$$V = \frac{1}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2) + \gamma(\xi_1\dot{\xi}_1 + \xi_2\dot{\xi}_2) + \frac{1}{2}(\nu^2 + 2b\gamma)(\xi_1^2 + \xi_2^2)$$

subject to the condition that $\gamma < 2b$. Its derivative with respect to the vector field, which is determined by Eqs (2.18), has the form

$$\frac{dV}{dt} = (-2b + \gamma)(\dot{\xi}_1^2 + \dot{\xi}_2^2) - \nu^2\gamma(\xi_1^2 + \xi_2^2) + 2b\Omega(-\xi_2\dot{\xi}_1 + \xi_1\dot{\xi}_2) + o(\dot{\boldsymbol{\xi}}^2 + \boldsymbol{\xi}^2)$$

The derivative dV/dt is negative-definite if the expression

$$\nu^2\gamma(2b - \gamma) - b^2\Omega^2$$

is positive and non-zero. Noting that

$$\nu^2\gamma(2b-\gamma)-b^2\Omega^2 \geq \nu^2\gamma(2b-\gamma)-b^2\mu^2\omega^2$$

and that the expression $\gamma(2b-\gamma)$ takes its maximum value when $\gamma = b$, we obtain the condition for to negative-definite dV/dt in the form $\nu^2 > \mu^2 > \omega^2$. Hence, $\omega_0 = \nu/\mu$.

Remark 3. The proposed approach may be found to be effective not only in the case of a periodic perturbation but, also, for any other oscillating perturbation which changes slowly with time and, in particular, for systems of the form (1.1) when ω is replaced by ε, where ε is a small parameter.

3. Equilibrium stability in the case of a high frequency perturbation

We will now consider a periodic perturbation with a high frequency ω. In this case, the stability of the equilibrium can be investigated under assumptions which are less restrictive with regard to the periodic perturbation and, in particular, the equations

$$\begin{aligned} &A_0\ddot{\mathbf{q}} + [D(\omega t)+G(\omega t)]\dot{\mathbf{q}} + [\Pi(\omega t)+C(\omega t)]\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \\ &\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C^{0,1,1}_{t\mathbf{q}\dot{\mathbf{q}}}(R\times D_{\mathbf{q}\dot{\mathbf{q}}}), \quad \|\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})\| = o(\|(\mathbf{q}, \dot{\mathbf{q}})\|) \end{aligned} \tag{3.1}$$

can be considered where $[D(\omega t)+G(\omega t)]$, $[\prod(\omega t)+C(\omega t)] \in C^0_t$. As above, we shall assume that the constant matrix A_0 is symmetric with positive eigenvalues and, furthermore, that

$$D = D^T, \quad G = -G^T, \quad \Pi = \Pi^T, \quad C = -C^T$$

We later use the notation

$$D^0 = \langle D\rangle = \frac{1}{T}\int_0^T D(\omega t)dt, \quad T = \frac{2\pi}{\omega}$$

and, similarly,

$$G^0 = \langle G\rangle, \quad \Pi^0 = \langle \Pi\rangle, \quad C^0 = \langle C\rangle$$

In addition, we represent the matrices being considered in the form

$$D = D^0 + D_1(\omega t), \quad G = G^0 + G_1(\omega t), \quad \Pi = \Pi^0 + \Pi_1(\omega t), \quad C = C^0 + C_1(\omega t)$$

where, correspondingly,

$$D_1 = D - D^0, \quad \langle D_1\rangle = 0, \ldots, C_1 = C - C^0, \quad \langle C_1\rangle = 0$$

Suppose $D_1^*(\omega t)$, $G_1^*(\omega t)$, $\prod_1^*(\omega t)$, $C_1^*(\omega t)$ are the primitives of the corresponding matrices $D_1(\omega t), \ldots, C_1(\omega t)$ such that

$$\langle D_1^*(\omega t)\rangle = \ldots = \langle C_1^*(\omega t)\rangle = 0$$

Since the primitives being considered contain the quantity ω^- with a multiplier, it is convenient to represent them later in the form

$$D_1^*(\omega t) = \omega^{-1}\tilde{D}_1(\omega t), \ldots, C_1^*(\omega t) = \omega^{-1}\tilde{C}_1(\omega t)$$

where $\tilde{D}_1(\omega t), \ldots, \tilde{C}_1(\omega t)$ are, as before, matrices with a zero mean.

Taking account of the notation introduced, we rewrite Eq. (3.1) in the form

$$A_0\ddot{\mathbf{q}} + (D^0+G^0)\dot{\mathbf{q}} + K(\omega t)\dot{\mathbf{q}} + (\Pi^0+C^0)\mathbf{q} + N(\omega t)\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \tag{3.2}$$

where

$$K(\omega t) = D_1(\omega t) + G_1(\omega t), \quad N(\omega t) = \Pi_1(\omega t) + C_1(\omega t)$$

Together with (3.2), we consider the truncated system

$$A_0\ddot{\mathbf{q}} + (D^0 + G^0)\dot{\mathbf{q}} + (\Pi^0 + C^0)\mathbf{q} = \mathbf{0} \tag{3.3}$$

and the characteristic equation corresponding to it

$$\left|A_0\lambda^2 + (D^0 + G^0)\lambda + (\Pi^0 + C^0)\right| = 0 \tag{3.4}$$

the roots of which are denoted by $\lambda_1, \ldots, \lambda_{2n}$.

Theorem 2. *Suppose the matrices* $A_0^{-1}[D(\omega t) + G(\omega t)]$ *and* $A_0^{-1}[\prod(\omega t) + C(\omega t)]$ *only contain the parameter* ω *as a multiplier accompanying* t.

Then a threshold value of the frequency $\omega = \omega_0$ exists such that, when $\omega > \omega_0$, the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = 0$ of system (3.1) is asymptotically stable if $\mathrm{Re}\lambda_i < 0$, where λ_i are the roots of Eq. (3.4) and, on the other hand, it is unstable if just a single root λ^* of Eq. (3.4) with a positive real part exists.

Proof. Let us assume that $\mathrm{Re}\lambda_i < 0$. Then, according to Lyapunov's theorem, a positive-definite quadratic form

$$V_1 = \frac{1}{2}\dot{\mathbf{q}}^T A_1\dot{\mathbf{q}} + \mathbf{q}^T B_1\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T F_1\mathbf{q}$$

exists in which A_1, B_1, F_1 are symmetric constant matrices which satisfy the equation

$$L(V_1) = U(\dot{\mathbf{q}}, \mathbf{q}) \tag{3.5}$$

where $U(\dot{\mathbf{q}}, \mathbf{q})$ is the negative-definite quadratic form

$$L(V_1) = (\dot{\mathbf{q}}^T A_1 + \mathbf{q}^T B_1)[-A_0^{-1}(D^0 + G^0)\dot{\mathbf{q}} - A_0^{-1}(\Pi^0 + C^0)\mathbf{q}] + \dot{\mathbf{q}}^T B_1\dot{\mathbf{q}} + \dot{\mathbf{q}}^T F_1\mathbf{q}$$

We select the function

$$V_1^* = V_1 + \omega^{-1}W_1 \tag{3.6}$$

as the Lyapunov function.Here,

$$W_1 = \dot{\mathbf{q}}^T A_0 A_0^{-1}\tilde{K}(\omega t)\dot{\mathbf{q}} + \mathbf{q}^T[B_1 A_0^{-1}\tilde{K}(\omega t) + \tilde{N}^T(\omega t)A_0^{-1}A_1]\dot{\mathbf{q}} + \mathbf{q}^T B_1 A_0^{-1}\tilde{N}(\omega t)\mathbf{q}$$

$$\tilde{K}(\omega t) = \tilde{D}_1(\omega t) + \tilde{G}_1(\omega t), \quad \tilde{N}(\omega t) = \tilde{\Pi}_1(\omega t) + \tilde{C}_1(\omega t)$$

Calculating the derivative of the function V_1^* with respect to t along the vector field defined by system (3.1), we obtain

$$\frac{dV_1^*}{dt} = U(\dot{\mathbf{q}}, \mathbf{q}) + \frac{1}{\omega}W_1^* + o(\|(\dot{\mathbf{q}}, \mathbf{q})\|^2) \tag{3.7}$$

where

$$W_1^* = -\mathbf{q}^T S_2 A_0^{-1}(\Pi + C)\mathbf{q} + \dot{\mathbf{q}}^T[S_2 - S_1 A_0^{-1}(D + G)]\dot{\mathbf{q}} +$$
$$+ \dot{\mathbf{q}}^T[S_3 - S_1 A_0^{-1}(\Pi + C) - (D - G)A_0^{-1}S_2^T]\mathbf{q}$$

$$S_1 = \tilde{K}^T A_0^{-1}A_1 + A_1 A_0^{-1}\tilde{K}, \quad S_2 = B_1 A_0^{-1}\tilde{K} + \tilde{N}^T A_0^{-1}A_1, \quad S_3 = B_1 A_0^{-1}\tilde{N} + \tilde{N}^T A_0^{-1}B_1$$

We now direct the parameter ω on the right-hand side of equalities (3.6) and (3.7) to infinity. Noting that the coefficients of the quadratic forms W_1 and W_1^* are bounded, we conclude that a threshold value ω_0 of the parameter ω exists such that, when $\omega > \omega_0$, the right-hand side of equality (3.6) becomes positive-definite and the right-hand side of equality (3.7) becomes negative-definite. Hence we conclude that Theorem 2 holds in the case when $\mathrm{Re}\lambda_i < 0$.

Now, suppose a root λ^* Eq. (3.4) exists such that $\mathrm{Re}\lambda^* > 0$. Then, according to Lyapunov's theorem, a quadratic form with alternating signs

$$V_2 = \frac{1}{2}\dot{\mathbf{q}}^T A_2 \dot{\mathbf{q}} + \mathbf{q}^T B_2 \dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T F_2 \mathbf{q}$$

exists which satisfies the equation

$$L(V_2) = \kappa V_2 + U_1(\dot{\mathbf{q}}, \mathbf{q}) \tag{3.8}$$

where κ is a positive constant, $U_1(\mathbf{q}, \dot{\mathbf{q}})$ is a positive-definite quadratic form, and we obtain the Lyapunov operator $L(V_2)$ from $L(V_1)$ by replacing the matrices A_1, B_1, F_1 by A_2, B_2, F_2.

We select as the Lyapunov function, the function

$$V_2^* = V_2 + \omega^{-1} W_2 \tag{3.9}$$

where we obtain W_2 from W_1 by replacing the matrices A_1, B_1 by A_2, B_2. Calculating the derivative of the function V_2^* with respect to t along the vector field defined by system (3.1), we have

$$\frac{dV_2^*}{dt} = \kappa V_2 + U_1(\dot{\mathbf{q}}, \mathbf{q}) + \omega^{-1} W_2^* + o(\|\dot{\mathbf{q}}, \mathbf{q}\|^2) \tag{3.10}$$

Here, as before, we obtain W_2^* from W_1^* by replacing the matrices A_1, B_1 by A_2, B_2.

We then rewrite equality (3.10) in the form

$$\frac{dV_2^*}{dt} = \kappa V_2^* + U_1(\dot{\mathbf{q}}, \mathbf{q}) + \omega^{-1}(-\kappa W_2 + W_2^*) + o(\|(\dot{\mathbf{q}}, \mathbf{q})\|^2) \tag{3.11}$$

The coefficients of the quadratic form $(-\kappa W_2 + W_2^*)$ are bounded regardless of the magnitude of ω, and, therefore, by subsequently following the scheme described above, we conclude that a threshold value ω_0 of the parameter ω exists such that, when $\omega > \omega_0$, the expression

$$U_1(\dot{\mathbf{q}}, \mathbf{q}) + \omega^{-1}(-\kappa W_2 + W_2^*)$$

becomes a positive-definite function and the quadratic form V_2^*, in accordance with expression (3.9), takes values with different signs. So, the conditions of the Chetayev instability theorem are satisfied in the situation being considered. □

Theorem 2 is proved.

Coarse estimates for ω can be obtained in a similar way to that indicated in Remark 1. We merely note that it is now necessary to obtain estimates of ω taking account of the fact that the functions V_1^* and V_2^* also contain the factor ω^{-1}.

Hence, under the conditions of Theorem 2, the qualitative properties of system (3.1) are entirely determined by the properties of truncated system (3.3), which is autonomous. In this sense, system (3.3), as the defining part of system (3.1), exhibits stiffness with respect to high-frequency periodic perturbations, which can be large in amplitude but, however, possess a zero mean.

Example 3. We will now consider the equations of a rotating arm which can be reduced to the form[13]

$$\ddot{\boldsymbol{\xi}} + 2(b+\mu)\dot{\boldsymbol{\xi}} + p^2\boldsymbol{\xi} + C(\omega t)\boldsymbol{\xi} = \mathbf{O}(\dot{\boldsymbol{\xi}}^2 + \boldsymbol{\xi}^2), \quad \boldsymbol{\xi} = (\xi_1, \xi_2)^T, \quad p^2 = \text{const} \tag{3.12}$$

where

$$C(\omega t) = \begin{pmatrix} 0 & 2b\Omega(\omega t) \\ -2b\Omega(\omega t) & 0 \end{pmatrix}, \quad \Omega(\omega t) = \Omega^0 + \Omega_1(\omega t), \quad \Omega^0 = \text{const}$$

$$\langle \Omega_1(\omega t) \rangle = 0$$

and the constant coefficients b and μ reflect the existence of internal and external resistance forces, and the quantity $\Omega(\omega t)$ denotes the angular velocity of rotation of the arm. We shall assume that the periodic function $\Omega_1(\omega t)$ contains only the parameter ω as a multiplier accompanying t.

We will investigate the stability of the solution $\dot{\boldsymbol{\xi}} = \boldsymbol{\xi} = \mathbf{0}$ of system (3.12).

In the case being considered, we have the system

$$\ddot{\boldsymbol{\xi}} + 2(b+\mu)\dot{\boldsymbol{\xi}} + p^2\boldsymbol{\xi} + C^0\boldsymbol{\xi} = \mathbf{0} \tag{3.13}$$

as an analogue of Eq. (3.3).

The matrix C^0 is obtained from the matrix $C(\omega t)$ by replacing $\Omega(\omega t)$ by Ω^0. The characteristic equation takes the form

$$\begin{vmatrix} \lambda^2 + 2(b+\mu)\lambda + p^2 & 2b\Omega^0 \\ -2b\Omega^0 & \lambda^2 + 2(b+\mu)\lambda + p^2 \end{vmatrix} = 0$$

Its roots have negative real parts if the inequality

$$(1+\mu/b)^2 p^2 > \Omega^{0^2} \tag{3.14}$$

is satisfied. We choose the Lyapunov function for the truncated system in the form (2.12)

$$V = \frac{1}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2) + \gamma(\xi_1\dot{\xi}_1 + \xi_2\dot{\xi}_2) + \frac{1}{2}[p^2 + 2(b+\mu)\gamma](\xi_1^2 + \xi_2^2)$$

Its derivative along the vector field determined by Eq. (3.13) has the form

$$\frac{dV}{dt} = [-2(b+\mu)+\gamma](\dot{\xi}_1^2 + \dot{\xi}_2^2) - \gamma p^2(\xi_1^2 + \xi_2^2) + 2b\Omega^0(-\xi_2\dot{\xi}_1 + \xi_1\dot{\xi}_2) \tag{3.15}$$

and is negative-definite if

$$\gamma p^2[2(b+\mu) - \gamma] - b^2\Omega^{0^2} > 0$$

Noting that the expression $\gamma[2(b+\mu) - \gamma]$ has a maximum value when $\gamma = b + \mu$, we obtain the condition for dV/dt to be negative-definite in the form of inequality (3.14). Hence, the function V can be considered as a solution of Lyapunov's equation of the type (3.5) in which the right-hand side of equality (3.15) plays in the role of $U(\dot{\mathbf{q}}, \mathbf{q})$.

In the case of the initial system (3.12), we select the function

$$V_1 = V - 2b\tilde{\Omega}_1\omega^{-1}(-\xi_2\dot{\xi}_1 + \xi_1\dot{\xi}_2)$$

as the Lyapunov function. We recall that, according to what has been stated above, $\Omega_1^*(\omega t) = \omega^{-1}\tilde{\Omega}_1(\omega t)$, where $\Omega_1^*(\omega t)$ is the primitive of the function $\Omega_1(\omega t)$ such that $\langle\Omega_1^*(\omega t) = 0\rangle$. The derivative of the function V_1 with respect to t along the vector field defined by system (3.12) has the form

$$\begin{aligned} \frac{dV_1}{dt} = &-(b+\mu)(\dot{\xi}_1^2 + \dot{\xi}_2^2) - [p^2(b+\mu) + 4b^2(\Omega^0 + \Omega_1)\omega^{-1}\tilde{\Omega}_1](\xi_1^2 + \xi_2^2) + \\ &+ [2b\Omega^0 + 4b(b+\mu)\omega^{-1}\tilde{\Omega}_1](-\xi_2\dot{\xi}_1 + \xi_1\dot{\xi}_2) + o(\dot{\boldsymbol{\xi}}^2 + \boldsymbol{\xi}^2) \end{aligned} \tag{3.16}$$

We now use the notation

$$\rho_1 = \max_{t\in[t,\,t+2\pi/\omega]} \Omega_1(\omega t), \quad \rho_2 = \max_{t\in[t,\,t+2\pi/\omega]} \tilde{\Omega}_1(\omega t)$$

The condition for the function V_1 to be positive-definite reduces to the inequality

$$p^2 + (b+\mu)^2 - 4b^2\rho_2^2\omega^{-2} > 0 \tag{3.17}$$

The derivative dV_1/dt is negative-definite if

$$\frac{1}{4}[(1+\mu/b)^2 p^2 - \Omega^{0^2}] - (b+\mu)\rho_1\rho_2\omega^{-1} - (b+\mu)^2\rho_2^2\omega^{-2} > 0 \tag{3.18}$$

On the basis of inequalities (3.17) and (3.18) we conclude that

$$1/\omega_0 = \min(1/\omega_0^{(1)}, 1/\omega_0^{(2)}) \tag{3.19}$$

where $1/\omega_0^{(1)}$ and $1/\omega_0^{(2)}$ are respectively the positive roots of the equations obtained by equating the left-hand sides of inequalities (3.17) and (3.18) to zero.

Starting out from these equations and inequality (3.19), there is no difficulty in determining the value of ω_0 for which the inequality $\omega > \omega_0$, together with inequality (3.14), ensures the asymptotic stability of the equilibrium being considered.

We note that, if, instead of the derivative of the function V_1 with respect to t along the vector field defined by system (3.12), the derivative of the function V with respect to t is considered, we obtain the condition for the asymptotic stability of the equilibrium in the form

$$(1+\mu/b)^2 p^2 > (\Omega^0 + \rho_1)^2$$

This condition is close to condition (3.14) and imposes a constraint on the amplitude of the periodic perturbation $\Omega_1(\omega t)$. If account is taken of inequality (3.14), the asymptotic stability conditions obtained using the function V_1 impose constraints on the mean of $\Omega(\omega t)$ and, also, on the frequency ω. Hence, according to these conditions, for any fixed amplitude of a periodic perturbation $\Omega_1(\omega t)$, it is possible to select a frequency ω such that the equilibrium position becomes asymptotically stable. There is nothing surprising about the fact that, depending on the structure of the auxiliary functions, we arrive at different versions of the conditions for asymptotic stability, since the topic of discussion is the sufficient conditions.

Remark 4. The proposed approach can prove to be useful not only in the case of periodic perturbations but also in the case of any other rapidly oscillating perturbation and, in particular, in the case of systems of the form

$$A_0\ddot{\mathbf{q}} + [D(\lambda t) + G(\lambda t)]\dot{\mathbf{q}} + [\Pi(\lambda t) + C(\lambda t)]\mathbf{q} = \mathbf{F}(\lambda t, \dot{\mathbf{q}}, \mathbf{q})$$

where λ is a large parameter. It is clear that, in this case, it is necessary to consider the question of the mean of the matrices $D(\lambda_t), \ldots, C(\lambda_t)$.

Remark 5. The requirement concerning the nature of the occurrence of the parameter ω in the equations of motion (3.1) in Theorem 2 is important. In order to demonstrate this, we will represent Eq. (3.1) in the form of the first order system

$$\dot{\mathbf{x}} = \mathbf{X}(\omega t, \mathbf{x}) \tag{3.20}$$

Assuming that the parameter ω is large and changing to fast time $\omega t = \tau$, we obtain

$$d\mathbf{x}/d\tau = \varepsilon\mathbf{X}(\tau, \mathbf{x}), \quad \varepsilon = \omega^{-1}$$

Hence, if the vector $\mathbf{X}(\omega t, \mathbf{x})$ in system (3.20) contains only the parameter ω as a multiplier accompanying t, we arrive at the equations in the standard Bogolyubov form in the case of sufficiently large ω. Without stipulations concerning the nature of the appearance of the parameter ω in the equations of motion, the standard form of the equations of motion cannot be obtained. The well-known Demidovich theorem on averaging contains such a stipulation, although in a somewhat different form.[14]

Returning to the well-known problem of the stabilization of the upper position of a pendulum, we will represent the corresponding second-order equation in the form of a system of two equations

$$\dot{x} = y, \quad \dot{y} = -\alpha y - (-b + a\omega^2 \sin\omega t)x + o(\|(x, y)\|) \tag{3.21}$$

where α, a and b are positive constants. In the case being considered, on changing to fast time $\omega t = \tau$ we obtain the system of equations

$$dx/d\tau = \omega^{-1}y, \quad dy/d\tau = \omega^{-1}(-\alpha y + bx) - (a\omega\sin\tau)x + o(\|(x, y)\|)$$

which is not a system in the standard Bogolyubov form. Additional transformations have been used[15] in order to reduce Eq. (3.21) to standard form. As a result, it has been shown that the equilibrium $x = y = 0$ is asymptotically stable for a sufficiently large value of ω.

Consequently, formal averaging in system (3.21) can lead to an erroneous result even in the case of large values of ω.

If the coefficients of the system contain the parameter ω, then additional constraints concerning the manner in which ω occurs in the equations of motion are necessary. This question has been discussed in detail in Ref. 9.

In the light of what has been stated above, within the framework of the approach considered it is not only a question of the proof of the existence of a threshold value ω_0 which ensures this or that property of the solutions of the system but, also, of the routes for determing this value in explicit form which, as the examples considered above show, is of fundamental importance when solving practical problems.

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